1 A natural definition for tempo ramping

1.1 Some definitions

The **time**, often denoted t, is the duration in some multiple unit of seconds (samples, superclock, etc).

We define b to be the **beats function**: b is the number of beats that occurred since $t = 0$ until the current time. It is assumed to be continuous and even derivable rather than being a staircase function: b will thus value 4.5 when exactly in the middle of beat 4, halfway[1] between pulsations 4 and 5.

The **tempo function**, or simply tempo, is the number T of pulsations per unit of time. Thus $T = \frac{\mathrm{d}b}{\mathrm{d}t}$.

A **tempo ramp** is a definition of T for $0 \leq t \leq \Delta t$ such that $T_{t=0} = T_0$ and $T_{t=\Delta t} = T_{\text{end}} = T_0 + \Delta T$ where Δt, T_0 and T_{end} or ΔT are given. Most of the times they are set by the user, but Δt can be sometimes defined in beats, that is be the unique Δt such that $b_{|t=\Delta t} = \Delta b$.

1.2 Linear tempo ramping

The simplest definition that comes to mind is a tempo ramp where the tempo increases from T_0 to $T_0 + \Delta T$ *linearly* with time.

Here $T = T_0 + \frac{\Delta T}{\Delta t} \times t$, and thus $b = \frac{\Delta T}{2\Delta t} \times t^2 + T_0 \times t$, assuming $b_{|t=0} = 0$.

If Δt is defined implicitly from a duration Δb in beats, then

$$\Delta b = \frac{\Delta T}{2\Delta t} \times \Delta t^2 + T_0 \Delta t = \left(\frac{1}{2}\Delta T + T_0\right) \times \Delta t$$

thus

$$\Delta t = \frac{2\Delta b}{\Delta T + 2T_0}$$

1.3 Exponential tempo ramping

Humans are not very good at keeping track precisely of absolute time for long durations, and are better at comparing short durations, that is maintain a reasonably stable time span between successive pulsations, and have a precise sense of elapsed time *in terms of these pulsations*.

In particular, when changing the tempo, it is less far-fetched to imagine a human feeling the tempo increase as a function of elapsed *beats* rather than absolute time in seconds which is a pulsation that is very hard to maintain constant and even sense when competing with the (increasing) musical pulsation.

In other words, a definition more suitable than that of linear ramping would be that the tempo increases *linearly* with the value of b. We thus want

$$T = T_0 + \frac{\Delta T}{\Delta b} \times b, \text{ that is } \frac{\mathrm{d}b}{\mathrm{d}t} - \frac{\Delta T}{\Delta b} \times b = T_0$$

The solutions of this order 1 linear differential equation are[2] $b = A\mathrm{e}^{\frac{\Delta T}{\Delta b}t} - \frac{\Delta b T_0}{\Delta T}$, and since we want $b_{|t=0} = 0$, we get $A = \frac{\Delta b T_0}{\Delta T}$ that is

$$b = \frac{\Delta b T_0}{\Delta T}(\mathrm{e}^{\frac{\Delta T}{\Delta b}t} - 1)$$

[1] Not necessarily halfway in time units, though.

[2] The exponential is the solution of the associated homogeneous equation and the second term is the constant solution of the equation.

Denoting $\omega = \dfrac{\Delta T}{\Delta b}$ we find $b = \frac{T_0}{\omega}(\mathrm{e}^{\omega t} - 1)$. (Note: ω is called c in Nick's text, but I like ω better since it is a frequency).

The *tempo* is $T = \dfrac{\mathrm{d}b}{\mathrm{d}t} = T_0 \mathrm{e}^{\frac{\Delta T}{\Delta b} t} = T_0 \mathrm{e}^{\omega t}$, and this is where the name exponential tempo ramp comes from.

Recall that Δb, ΔT and T_0 are parameters of the ramp, while t is the arbitrary time within the ramp at which we want to know b and T.

Let us compute the reciprocal of b:

$$b = \frac{\Delta b T_0}{\Delta T}(\mathrm{e}^{\frac{\Delta T}{\Delta b} t} - 1) \iff \frac{\Delta T}{\Delta b T_0} b = \mathrm{e}^{\frac{\Delta T}{\Delta b} t} - 1 \iff 1 + \frac{\Delta T}{\Delta b T_0} b = \mathrm{e}^{\frac{\Delta T}{\Delta b} t}$$

$$\iff \log\left(1 + \frac{\Delta T}{\Delta b T_0} b\right) = \frac{\Delta T}{\Delta b} t \iff t = \frac{\Delta b}{\Delta T} \log\left(1 + \frac{\Delta T}{\Delta b T_0} b\right)$$

If Δt is defined directly instead of implicitly from a duration Δb in beats, then we have to find out the value of Δb. At $t = \Delta t$, we have $b = \Delta b$ thus

$$\Delta t = \frac{\Delta b}{\Delta T} \log\left(1 + \frac{\Delta T}{\Delta b T_0} \Delta b\right) \iff \Delta t = \frac{\Delta b}{\Delta T} \log\left(1 + \frac{\Delta T}{T_0}\right) \iff \Delta b = \frac{\Delta T \Delta t}{\log\left(1 + \frac{\Delta T}{T_0}\right)}$$

A more useful expression is

$$\omega = \frac{\Delta T}{\Delta b} = \frac{1}{\Delta t} \log\left(1 + \frac{\Delta T}{T_0}\right)$$

which gives

$$b = \frac{\Delta t T_0}{\log\left(1 + \frac{\Delta T}{T_0}\right)} \mathrm{e}^{\frac{t}{\Delta t} \log\left(1 + \frac{\Delta T}{T_0}\right)} = \frac{\Delta t T_0}{\log\left(1 + \frac{\Delta T}{T_0}\right)} \left(1 + \frac{\Delta T}{T_0}\right)^{\frac{t}{\Delta t}}$$

and

$$T = T_0 \mathrm{e}^{\frac{t}{\Delta t} \log\left(1 + \frac{\Delta T}{T_0}\right)} = T_0 \left(1 + \frac{\Delta T}{T_0}\right)^{\frac{t}{\Delta t}}$$

We at last compute the reciprocals:

$$b = \frac{\Delta t T_0}{\log\left(1 + \frac{\Delta T}{T_0}\right)} \mathrm{e}^{\frac{t}{\Delta t} \log\left(1 + \frac{\Delta T}{T_0}\right)} \iff \frac{\log\left(1 + \frac{\Delta T}{T_0}\right)}{\Delta t T_0} b = \mathrm{e}^{\frac{t}{\Delta t} \log\left(1 + \frac{\Delta T}{T_0}\right)}$$

$$\iff \log\left(\frac{\log\left(1 + \frac{\Delta T}{T_0}\right)}{\Delta t T_0} b\right) = \frac{t}{\Delta t} \log\left(1 + \frac{\Delta T}{T_0}\right) \iff t = \frac{\Delta t \log\left(\frac{\log\left(1 + \frac{\Delta T}{T_0}\right)}{\Delta t T_0} b\right)}{\log\left(1 + \frac{\Delta T}{T_0}\right)}$$

1.4 Summary with T as main tempo representation

The following table gathers all results. For code factorization we might want to only consider the red formulas, that is compute (and maybe cache) ω beforehand. Note that in the formulas, ω is *always* the inverse of a time, never in per-beats.

When you know	Δt	Δb	ω
Compute ω	$\frac{1}{\Delta t}\log\left(1+\frac{\Delta T}{T_0}\right)$	$\frac{\Delta T}{\Delta b}$	
Compute b from t	$\frac{\Delta t T_0}{\log\left(1+\frac{\Delta T}{T_0}\right)}\left(1+\frac{\Delta T}{T_0}\right)^{\frac{t}{\Delta t}}$	$\frac{\Delta b T_0}{\Delta T}(\mathrm{e}^{\frac{\Delta T}{\Delta b}t}-1)$	$\frac{T_0}{\omega}(\mathrm{e}^{\omega t}-1)$
Compute T from t	$T_0\left(1+\frac{\Delta T}{T_0}\right)^{\frac{t}{\Delta t}}$	$T_0\mathrm{e}^{\frac{\Delta T}{\Delta b}t}$	$T_0\mathrm{e}^{\omega t}$
Compute t from b	$\frac{\Delta t\log\left(\frac{\log\left(1+\frac{\Delta T}{T_0}\right)}{\Delta t T_0}b\right)}{\log\left(1+\frac{\Delta T}{T_0}\right)}$	$\frac{\Delta b}{\Delta T}\log\left(1+\frac{\Delta T}{\Delta b T_0}b\right)$	$\frac{1}{\omega}\log\left(1+\frac{\omega b}{T_0}\right)$
Compute T from b	$T_0+\frac{1}{\Delta t}\log\left(1+\frac{\Delta T}{T_0}\right)\times b$	$T_0+\frac{\Delta T}{\Delta b}\times b$	$T_0+\omega b$

Note that there are often library functions to compute $\log(1+x)$ directly from x more precise than computing $1+x$ then its logarithm. The main reason is that when adding 1to some very small number you loose a lot of precision because the exponent is now tailored to the representation of 1, and that the Taylor-McLaurin series for $\log(1+x)$ is a very efficient mean to compute an approximate value for it.

1.5 Using $S = \frac{1}{T}$ to represent tempo

Since the base time unit in ardour is not seconds but far smaller than that[3], any value of T or ΔT will be very small. A better value to store, that can easily be rounded to integer without a huge loss of precision, is $S = 1/T$.

Instead of having T_0 and $T_1 = T_0 + \Delta T$, we thus assume that when computing the formulas we have access to S_0 and S_1.

Then from Δt, $\omega = \frac{1}{\Delta t}\log\left(1+\frac{\Delta T}{T_0}\right) = \frac{1}{\Delta t}\log\left(\frac{T_1}{T_0}\right) = \frac{1}{\Delta t}\log\left(\frac{S_0}{S_1}\right)$.

From beats, $\omega = \frac{\Delta T}{\Delta b} = \frac{S_0 - S_1}{S_0 S_1 \Delta b}$ (or just use $\omega = \left(\frac{1}{S_1} - \frac{1}{S_0}\right)/\Delta b$).

To compute b from t we can use $b = \frac{\mathrm{e}^{\omega t} - 1}{S_0 \omega}$.

And for S from t we have $S = \frac{S_0}{\mathrm{e}^{\omega t}} = S_0 \mathrm{e}^{-\omega t}$.

As for t from b the formula becomes $t = \frac{1}{\omega}\log(1 + S_0 \omega b)$.

Lastly, for S from b we can see:

$$S = 1/T = \frac{1}{1/S_0 + \omega b} = \frac{S_0}{1 + S_0 \omega b}$$

Maybe we can also to compute and cache $\omega' = S_0\omega = \frac{S_0}{\Delta t}\log\left(\frac{S_0}{S_1}\right) = \frac{S_0 - S_1}{S_1 \Delta b}$ Which is in « per-beats » and occurs often in the formulas (but not exclusively). Note that the fact that ω or ω' is relevant is *not* depending on whether we compute from time or from beats, or even if the ramp length is defined in time or in beats.

[3] It was samples in previous versions of Ardour, and is even more precise in later versions.